s equation for $\dot{\psi}$ and substituting into Equation 7.68, we obtain

$$\begin{aligned} v^2 &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1 - \varepsilon^2 \cos^2 \psi}{(1 - \varepsilon \cos \psi)^2} \\ &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1 + \varepsilon \cos \psi}{1 - \varepsilon \cos \psi} \\ &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{2 - (1 - \varepsilon \cos \psi)}{1 - \varepsilon \cos \psi} \end{aligned} \tag{7.70}$$

g $r/a = 1 - \varepsilon \cos \psi$ from Equation 7.58, there results

$$v^2 = \left(\frac{2\pi}{\tau}\right)^2 a^3 \left(\frac{2}{r} - \frac{1}{a}\right) \tag{7.71}$$

pler's Third Law (Equation 7.48) can be used to reduce this expres-

$$\boxed{v^2 = \frac{k}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{7.72}$$

vish to calculate $\theta(t)$ for the motion of a body whose orbit has an not too large (say, $\varepsilon \approx 0.1$), and if we wish to achieve an accuracy of, in 10^6, then many terms in Equation 7.54 are necessary. The use of uation in such a situation is somewhat easier. Astronomical cal- orbits are almost always based on Kepler's equation. Details of ion procedures for Kepler's equation can be found in various texts on chanics.*

idal Angles and Precession (optional)

executes bounded, noncircular motion in a central-force field, then istance from the force center to the particle must always be in the $\geq r \geq r_{min}$, that is, r must be bounded by the apsidal distances. Fig- cates that only *two* apsidal distances exist for bounded, noncircular t in executing one complete revolution in θ, the particle may not original position (see Figure 7-4). The angular separation between ive values of $r = r_{max}$ depends on the exact nature of the force. The en any two consecutive apsides is called the **apsidal angle**, and, since a must be symmetrical about any apsis, it follows that all apsidal uch motion must be equal. The apsidal angle for elliptical motion,

*0 methods of obtaining approximate solutions to Kepler's equation are discussed in See, for example, Moulton (Mo58, p. 164ff) for details.

for example, is just π. If the orbit is not closed, the particle reaches the apsidal distances at different points in each revolution; the apsidal angle is not then a rational fraction of 2π, as is required for a closed orbit. If the orbit is *almost* closed, the apsides *precess*, or rotate slowly in the plane of the motion. This effect is exactly analogous to the slow rotation of the elliptical motion of a two-dimensional harmonic oscillator whose natural frequencies for the x- and y-motions are almost equal (see Section 3.4).

Since an inverse-square-law force requires that all elliptical orbits be exactly closed, the apsides must stay fixed in space for all time. If the apsides move with time, however slowly, this indicates that the force law under which the body moves does not vary exactly as the inverse square of the distance. This important fact was realized by Newton, who pointed out that any advance or regression of a planet's perihelion would require the radial dependence of the force law to be slightly different from $1/r^2$. Thus, Newton argued, the observation of the time dependence of the perihelia of the planets would be a sensitive test of the validity of the form of the universal gravitation law.

In point of fact, for planetary motion within the solar system, one expects that, because of the perturbations introduced by the existence of all of the other planets, the force experienced by any planet does not vary exactly as $1/r^2$, if r is measured from the sun. This effect is small, however, and only slight variations of planetary perihelia have been observed. The perihelion of Mercury, for example, which shows the largest effect, advances only about 574″ of arc per century.* Detailed calculations of the influence of the other planets on the motion of Mercury predict that the rate of advance of the perihelion should be approximately 531″ per century. The uncertainties in this calculation are considerably less than the difference of 43″ between observation and calculation,† and for a considerable time this discrepancy was the outstanding unresolved difficulty in the Newtonian theory. We now know that the modification introduced into the equation of motion of a planet by the general theory of relativity almost exactly accounts for the difference of 43″. This result is one of the major triumphs of relativity theory.

We next indicate the way the advance of the perihelion can be calculated from the modified equation of motion. To perform this calculation, it is convenient to use the equation of motion in the form of Equation 7.20. If we use

*This precession is in addition to the general precession of the equinox with respect to the "fixed" stars, which amounts to 5025.645″ ± 0.050″ per century.

†In 1845, the French astronomer Urbain Jean Joseph Le Verrier (1811–1877) first called attention to the irregularity in the motion of Mercury. Similar studies by LeVerrier and by the English astronomer John Couch Adams of irregularities in the motion of Uranus led to the discovery of the planet Neptune in 1846. An interesting account of this episode is given by Turner (Tu04, Chapter 2). We must note, in this regard, that perturbations may be either *periodic* or *secular* (i.e., ever increasing with time). Laplace showed in 1773 (published, 1776) that any perturbation of a planet's mean motion that is caused by the attraction of another planet must be periodic, although the period may be extremely long. This is the case for Mercury; the precession of 531″ per century is periodic, but the period is so long that the change from century to century is small compared to the residual effect of 43″.

al gravitational law for $F(r)$, we can write

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{1}{u^2}F(u)$$
$$= \frac{Gm^2M}{l^2} \tag{7.73}$$

onsider the motion of a body of mass m in the gravitational field of a ass M. The quantity u is therefore the reciprocal of the distance and M.

nodification of the gravitational force law required by the general elativity introduces into the force a small component that varies as Thus we have

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm^2M}{l^2} + \frac{3GM}{c^2}u^2 \tag{7.74}$$

the velocity of propagation of the gravitational interaction and is vith the velocity of light.* To simplify the notation, we define

$$\frac{1}{\alpha} \equiv \frac{Gm^2M}{l^2}$$
$$\delta \equiv \frac{3GM}{c^2} \tag{7.75}$$

write Equation 7.74 as

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} + \delta u^2 \tag{7.76}$$

onlinear equation, and we use a successive approximation procedure solution. We choose the first trial solution to be the solution of Equa- the case that the term δu^2 is neglected†:

$$u_1 = \frac{1}{\alpha}(1 + \varepsilon\cos\theta) \tag{7.77}$$

familiar result for the pure inverse-square-law force (see Equa- Note that α is here the same as that defined in Equation 7.40 except

he relativistic term results from effects understandable in terms of special relativity, viz., 1/3) and the relativistic momentum effect (1/6); the velocity is greatest at perihelion and on (see Chapter 14). The other half of the term arises from general relativistic effects and with the finite propagation time of gravitational interactions. Thus the agreement y and experiment confirms the prediction that the gravitational propagation velocity is at for light.

the necessity of introducing an arbitrary phase into the argument of the cosine term by neasure θ from the position of perihelion; i.e., u_1 is a maximum (and hence r_1 is a $\theta = 0$.

that μ has been replaced by m. If we substitute this expression into the right-hand side of Equation 7.76 we find

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} + \frac{\delta}{\alpha^2}[1 + 2\varepsilon\cos\theta + \varepsilon^2\cos^2\theta]$$
$$= \frac{1}{\alpha} + \frac{\delta}{\alpha^2}\left[1 + 2\varepsilon\cos\theta + \frac{\varepsilon^2}{2}(1 + \cos 2\theta)\right] \tag{7.78}$$

where $\cos^2\theta$ has been expanded in terms of $\cos 2\theta$. The first trial function u_1, when substituted into the left-hand side of Equation 7.76, reproduces only the first term on the right-hand side: $1/\alpha$. We can therefore construct a second trial function by adding to u_1 a term that reproduces the remainder of the right-hand side (in Equation 7.78). We can verify that such a particular integral is

$$u_p = \frac{\delta}{\alpha^2}\left[\left(1 + \frac{\varepsilon^2}{2}\right) + \varepsilon\theta\sin\theta - \frac{\varepsilon^2}{6}\cos 2\theta\right] \tag{7.79}$$

The second trial function is therefore

$$u_2 = u_1 + u_p$$

If we stop the approximation procedure at this point, we have

$$u \cong u_2 = u_1 + u_p$$
$$= \left[\frac{1}{\alpha}(1 + \varepsilon\cos\theta) + \frac{\delta\varepsilon}{\alpha^2}\theta\sin\theta\right]$$
$$+ \left[\frac{\delta}{\alpha^2}\left(1 + \frac{\varepsilon^2}{2}\right) - \frac{\delta\varepsilon^2}{6\alpha^2}\cos 2\theta\right] \tag{7.80}$$

where we have regrouped the terms in u_1 and u_p.

Consider the terms in the second set of brackets in Equation 7.80: the first of these is just a constant, and the second is only a small and periodic disturbance of the normal Keplerian motion. Therefore, on a long time scale neither of these terms contributes, on the average, to any change in the positions of the apsides. But in the first set of brackets, the term proportional to θ produces secular and therefore observable effects. Let us consider the first set of brackets:

$$u_{\text{secular}} = \frac{1}{\alpha}\left[1 + \varepsilon\cos\theta + \frac{\delta\varepsilon}{\alpha}\theta\sin\theta\right] \tag{7.81}$$

Next we can expand the quantity

$$1 + \varepsilon\cos\left(\theta - \frac{\delta}{\alpha}\theta\right) = 1 + \varepsilon\left(\cos\theta\cos\frac{\delta}{\alpha}\theta + \sin\theta\sin\frac{\delta}{\alpha}\theta\right)$$
$$\cong 1 + \varepsilon\cos\theta + \frac{\delta\varepsilon}{\alpha}\theta\sin\theta \tag{7.82}$$

have used the fact that δ is small to approximate

$$\cos\frac{\delta}{\alpha}\theta \cong 1, \qquad \sin\frac{\delta}{\alpha}\theta \cong \frac{\delta}{\alpha}\theta$$

can write u_{secular} as

$$u_{\text{secular}} \cong \frac{1}{\alpha}\left[1 + \varepsilon\cos\left(\theta - \frac{\delta}{\alpha}\theta\right)\right] \tag{7.83}$$

ave chosen to measure θ from the position of perihelion at $t = 0$. appearances at perihelion result when the argument of the cosine term ncreases to $2\pi, 4\pi, \ldots$, and so forth. But an increase of the argument by s that

$$\theta - \frac{\delta}{\alpha}\theta = 2\pi$$

$$\theta = \frac{2\pi}{1 - (\delta/\alpha)} \cong 2\pi\left(1 + \frac{\delta}{\alpha}\right)$$

the effect of the relativistic term in the force law is to displace the in each revolution by an amount

$$\Delta \cong \frac{2\pi\delta}{\alpha} \tag{7.84a}$$

apsides rotate slowly in space. If we refer to the definitions of α and δ s 7.75), we find

$$\Delta \cong 6\pi\left(\frac{GmM}{cl}\right)^2 \tag{7.84b}$$

ations 7.40 and 7.42 we can write $l^2 = \mu ka(1 - \varepsilon^2)$; then, since $k =$ $\mu \cong m$, we have

$$\boxed{\Delta \cong \frac{6\pi GM}{ac^2(1 - \varepsilon^2)}} \tag{7.84c}$$

refore that the effect is enhanced if the semimajor axis a is small and if ricity is large. Mercury, which is the planet nearest the sun and which ost eccentric orbit of any planet (except Pluto), provides the most est of the theory.* The calculated value of the precessional rate for

y, we can say that the relativistic advance of the perihelion is a maximum for Mercury rbital velocity is greatest for Mercury and the relativistic parameter v/c largest (see nd Problem 14-35).

Table 7-2
PRECESSIONAL RATES FOR THE PERIHELIA OF SOME PLANETS

	Precessional rate (seconds of arc/century)	
Planet	**Calculated**	**Observed**
Mercury	43.03 ± 0.03	43.11 ± 0.45
Venus	8.63	8.4 ± 4.8
Earth	3.84	5.0 ± 1.2
Mars	1.35	—
Jupiter	0.06	—

Mercury is $43.03'' \pm 0.03''$ of arc per century. The observed value (corrected for the influence of the other planets) is $43.11'' \pm 0.45''$,* so the prediction of relativity theory is confirmed in striking fashion. The precessional rates for some of the planets are given in Table7-2.

7.10 Stability of Circular Orbits

In Section 7.6 we pointed out that the orbit is circular if the total energy equals the minimum value of the effective potential energy, $E = V_{\text{min}}$. More generally, however, a circular orbit is allowed for *any* attractive potential, since the attractive force can *always* be made to just balance the centrifugal force by the proper choice of radial velocity. Although circular orbits are therefore always possible in a central, attractive force field, such orbits are not necessarily stable. A circular orbit at $r = \rho$ exists if $\dot{r}|_{r=\rho} = 0$ for all t; this is possible if $(\partial V/\partial r)|_{r=\rho} = 0$. But only if the effective potential has a *true minimum* does stability result. All other equilibrium circular orbits are unstable.

Let us consider an attractive central force with the form

$$F(r) = -\frac{k}{r^n} \tag{7.85}$$

The potential function for such a force is

$$U(r) = -\frac{k}{n-1}\cdot\frac{1}{r^{(n-1)}} \tag{7.86}$$

and the effective potential function is

$$V(r) = -\frac{k}{n-1}\cdot\frac{1}{r^{(n-1)}} + \frac{l^2}{2\mu r^2} \tag{7.87}$$

*R. L. Duncombe, *Astron. J. 61*, 174 (1956); see also G. M. Clemence, *Rev. Mod. Phys. 19*, 361 (1947).