231(3): Generalization of the Einstein Energy Equation using the Concept of Tetrad as Metric

The fundamental equation derived in note 231(2) is:

$$q^a_\mu = g^a_\mu = g^{a\nu} g_{\mu\nu} \quad - (1)$$

The tetrad is therefore a mixed index metric linking a and μ. Here a and μ can be two different representations of the same space, or two different mathematical spaces.

If eq. (1) is written out in full:

$$q^a_\mu = g^a_\mu = \begin{bmatrix} g^{(1)1} & g^{(1)2} \\ g^{(2)1} & g^{(2)2} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} - (2)$$

where the transverse components are used for illustration.

In flat 3-D space:

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - (3)$$

so in flat space:

$$q^a_\mu = g^{a\nu} g_{\mu\nu} \quad - (4)$$

$$= g^{a\nu} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so

$$g^{(1)}_1 = g^{(1)1} g_{11} \quad - (5)$$

$$g^{(1)}_2 = g^{(1)2} g_{22} \quad - (6)$$

$$g^{(2)}_{1} = g^{(2)1} g_{11} \quad -(7)$$

$$g^{(2)}_{2} = g^{(2)2} g_{22} \quad -(8)$$

If (a) denotes the circular polar basis:

$$g^{(1)}_{1} = g^{(1)1} = \underline{e}^{(1)} \cdot \underline{i} \quad -(9)$$

$$g^{(1)}_{2} = g^{(1)2} = \underline{e}^{(1)} \cdot \underline{j} \quad -(10)$$

$$g^{(2)}_{1} = g^{(2)1} = \underline{e}^{(2)} \cdot \underline{i} \quad -(11)$$

$$g^{(2)}_{2} = g^{(2)2} = \underline{e}^{(2)} \cdot \underline{j} \quad -(12)$$

In general:

$$V^{a} = q^{a}_{\mu} V^{\mu} \quad -(13)$$

i.e

$$V^{a} = g^{a}_{\mu} V^{\mu}, \quad -(14)$$

$$V^{a} = g^{a\nu} g_{\mu\nu} V^{\mu} = g^{a\nu} V_{\nu} \quad -(15)$$

Therefore the Cartan tetrad can be defined by:

$$\boxed{V^{a} = g^{a\nu} V_{\nu}} \quad -(16)$$

in terms of the mixed index metric $g^{a\nu}$. The latter is defined by:

$$\boxed{g^{a\nu} = h^{a} h^{\nu} \underline{e}^{a} \cdot \underline{e}^{\nu}} \quad -(17)$$

and is symmetric:

3)

$$g^{a\nu} = g^{\nu a} \quad - (18)$$

In eq. (17), h^a and h^ν are scaling factors and $\underline{e}^a$ and $\underline{e}^\nu$ are unit vectors. Eq. (17) is a generalization of eqs. (9) to (12). The unit vector of Minkowski spacetime in the Cartesian representation is:

$$\underline{e}^\nu = (e^0, \underline{i}, \underline{j}, \underline{k}) \quad - (19)$$

and in the circular polar representation:

$$\underline{e}^{(a)} = (e^{(0)}, \underline{e}^{(1)}, \underline{e}^{(2)}, \underline{e}^{(3)}) \quad - (20)$$

where

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}) \quad - (21)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}) \quad - (21a)$$

In this case: $h^{(a)} = h^\nu = 1 \quad - (22)$

and

$$\boxed{g^{(a)\nu} = \underline{e}^{(a)} \cdot \underline{e}^\nu} \quad - (23)$$

All the equations of differential geometry can be developed with the fundamental definition (23). If $\underline{e}^\nu$ is defined by eq. (19) then all the dynamical information is defined by $\underline{e}^{(a)}$. For example phases can be added to eqs. (21):

4)

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i\phi} \quad -(24)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi} \quad -(25)$$

For the purposes of this note, write the Einstein energy equation as:

$$p^\mu p_\mu = q^a_\mu p_a p^\mu \quad -(26)$$

or:

$$p^a p_\mu = q^a_\mu p^\mu p_\mu = q^a_\mu m^2 c^2$$

$$= g^{a\nu} g_{\mu\nu} m^2 c^2 \quad -(27)$$

From eq. (23):

$$\boxed{\frac{p^a p_\mu}{m^2 c^2} = \underline{e}^a \cdot \underline{e}^\nu g_{\mu\nu}} \quad -(28)$$

which generalizes the Einstein energy equation:

$$p^\mu p_\mu = m^2 c^2 \quad -(29)$$

Eq. (28) is one of general relativity, while eq. (29) is one of special relativity.