229(6): Absorption and Quantum Tunnelling

Consider the ECE wave equation:

$$(\square + R)\, q^a_\mu = 0, \quad -(1)$$

which can be expanded as:

$$(\square + R)\begin{bmatrix} \phi^R_1 & \phi^R_2 \\ \phi^L_1 & \phi^L_2 \end{bmatrix} = 0 \quad -(2)$$

and reduced to the fermion equation. Eq. (2) can also be written as the Klein Gordon equation:

$$\left(\square + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(3)$$

for a free particle. Eq. (3) can be deduced from the Einstein energy equation:

$$E^2 = p^2c^2 + m^2c^4 \quad -(4)$$

using:

$$E = i\hbar\frac{\partial}{\partial t}, \quad \underline{p} = -i\hbar\underline{\nabla}, \quad -(5)$$

so:

$$\left(\frac{mc}{\hbar}\right)^2 = \left(\frac{\omega}{c}\right)^2 - \kappa^2, \quad -(6)$$

and

$$\left(\square + \frac{\omega^2}{c^2} - \kappa^2\right)\phi = 0. \quad -(7)$$

Eq. (7) can also be written as:

$$\boxed{\begin{aligned}(E^2 - c^2p^2)\phi &= m^2c^4\phi \quad -(8)\\ &= \hbar^2(\omega^2 - \kappa^2c^2)\phi\end{aligned}}$$

2) Eq. (8) is an example of wave particle dualism and can be linearized using:

$$E^2 - c^2 p^2 = (E + cp)(E - cp). \quad -(9)$$

In the SU(2) basis:

$$\left(E^2 - c^2 \underline{\sigma} \cdot \underline{p}\, \underline{\sigma} \cdot \underline{p}\right) \phi = m^2 c^4 \phi. \quad -(10)$$

In these equations the quantum of spacetime energy is $\hbar\omega$ and the quantum of spacetime momentum is $\hbar \underline{\kappa}$. In eq. (8) they represent the energy and momentum quanta of a matter wave, whose mass is:

$$m = \frac{\hbar}{c^2}\left(\omega^2 - \kappa^2 c^2\right)^{1/2} \quad -(11)$$

Eq. (8) can be reduced to a Schroedinger equation using:

$$(E - cp)\phi = \frac{m^2 c^4}{E + cp}\phi \quad -(12)$$

or

$$\left(E^2 - m^2 c^4\right)\phi = c^2 p^2 \phi, \quad -(13)$$

$$E^2 - m^2 c^4 = (E - mc^2)(E + mc^2). \quad -(14)$$

Therefore:

3) $$(E - mc^2)\phi = \left(\frac{c^2 p^2}{E + mc^2}\right)\phi. \quad -(15)$$

In the non-relativistic approximation:

$$(E - mc^2)\phi = \frac{1}{2m} p^2 \phi, \quad -(16)$$

i.e

$$E \rightarrow mc^2. \quad -(17)$$

Eq. (16) is the free particle Schroedinger equation, usually written as:

$$-\frac{\hbar^2}{2m}\nabla^2\phi = E\phi \quad -(18)$$

i.e the energy E is by implication the energy

$$E - E_0 = E - mc^2, \quad -(19)$$

So

$$\nabla^2\phi = -\frac{2mE}{\hbar^2}\phi \quad -(20)$$

where

$$m = \frac{\hbar}{c^2}\left(\omega^2 - \kappa^2 c^2\right)^{1/2}. \quad -(21)$$

In eq. (21)

$$E = \hbar\omega = \gamma mc^2 \quad -(22)$$

$$\underline{p} = \hbar\underline{\kappa} = \gamma m\underline{v} \quad -(23)$$

4) In the free particle Schroedinger equation (20) the process of absorption of spacetime energy and momentum can be described as a change of mass:

$$m \rightarrow m + m_1 \quad - (24)$$

$$= \frac{\hbar}{c^2}\left[(\omega^2 - \kappa^2 c^2)^{1/2} + (\omega_1^2 - \kappa_1^2 c^2)^{1/2}\right]$$

so the effective mass of the particle increases. The same conclusion holds for the reduced mass of two interacting particles.

In the presence of a potential V, eq. (18) becomes:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi \quad - (25)$$

so

$$\nabla^2\psi = -\frac{2m}{\hbar^2}(E - V)\psi \quad - (26)$$

In the WKB approximation, the transmission coefficient for quantum tunnelling from eq. (26) is:

$$T = \frac{4}{\left(2\theta + \frac{1}{2\theta}\right)^2} \quad - (27)$$

) where for any V:

$$\theta = \exp\left(\frac{(2\mu)^{1/2}}{\hbar}\int_a^b \left(V(x)-E\right)^{1/2} dx\right) \quad -(28)$$

If the tunnelling process is accompanied by absorption, then

$$\boxed{\mu \rightarrow \mu + \frac{\hbar}{c^2}\left(\omega_1^2 - \kappa_1^2 c^2\right)^{1/2}} \quad -(29)$$

where ω_1 is the angular frequency and $\underline{\kappa_1}$ the wavenumber of the incoming wave. In some theories this is described as a phonon wave