

## 228(7): Relativistic Generalization of Quantum Tunneling Theory.

Consider the Einstein energy equation:

$$E^2 = \gamma^2 m^2 c^4 = p^2 c^2 + m^2 c^4, \quad - (1)$$

So:

$$\gamma m c^2 = \frac{1}{\gamma m} (p^2 + m^2 c^2) \quad - (2)$$

This is the necessary linearization needed for the quantum theory, i.e.:

$$\hat{H} \psi = E \psi \quad - (3)$$

where

$$\hat{H} = \frac{1}{\gamma m} (p^{\wedge 2} + m^2 c^2) \quad - (4)$$

$$E = \gamma m c^2 \quad - (5)$$

It follows that:

$$p^{\wedge 2} \psi = m^2 c^2 (\gamma^2 - 1) \psi \quad - (6)$$

The basic postulate of quantum mechanics is:

$$p^{\wedge n} = i \hbar \partial^{\wedge n} = \hbar \kappa^{\wedge n} \quad - (7)$$

where:

$$2) \quad p^\mu = \left( \frac{E}{c}, \underline{p} \right); \quad \partial^\mu = \left( \frac{1}{c} \frac{\partial}{\partial t}, -\underline{\nabla} \right); \quad \kappa^\mu = \left( \frac{\omega}{c}, \underline{\kappa} \right) \quad - (8)$$

So:

$$\hat{p}^2 \psi = \hbar^2 \kappa^2 \psi = m^2 c^2 (\gamma^2 - 1) \psi = \left( \frac{E^2}{c^2} - m^2 c^2 \right) \psi$$

- (9)

For a free particle:

$$\kappa = \frac{m c}{\hbar} (\gamma^2 - 1)^{1/2} \quad - (10)$$

In the non-relativistic limit:

$$m^2 c^2 (\gamma^2 - 1) = m^2 c^2 \left( \left( 1 - \frac{v^2}{c^2} \right)^{-1} - 1 \right)$$

$$\xrightarrow{v \ll c} m^2 v^2 \quad - (11)$$

so

$$p \rightarrow m v \quad - (12)$$

For the purpose of quantum tunnelling theory denote:

$$\hbar \kappa = \frac{m c}{\hbar} (\gamma^2 - 1)^{1/2} \quad - (13)$$

In the presence of a potential  $\bar{V}_0$ ,

3) eq. (3) becomes:

$$\hat{H}\psi = E\psi \quad - (14)$$

where

$$\hat{H} = \frac{1}{\gamma_m} (\hat{p}^2 + m^2 c^2) + V \quad - (15)$$

so

$$\begin{aligned} \hat{p}^2 \psi &= (\gamma_m (E - V) - m^2 c^2) \psi \\ &= \hbar^2 \kappa^2 \psi \quad - (16) \end{aligned}$$

and

$$\boxed{\kappa^2 = \frac{1}{\hbar^2} (\gamma_m (E - V) - m^2 c^2)} \quad - (17)$$

Denote

$$\kappa_0 = \left( \frac{mc}{\hbar} \right)^2 \quad - (18)$$

This is the Compton wave number. So:

$$\boxed{\kappa^2 + \kappa_0^2 = \frac{\gamma_m (E - V)}{\hbar^2}} \quad - (19)$$

Eq. (13) is:

$$\hbar^2 k^2 = \left( \frac{mc}{\hbar} \right)^2 (\gamma^2 - 1) \quad - (20)$$

i.e.

$$4) \quad p^2 + \kappa_0^2 = \gamma^2 \left( \frac{mc}{\hbar} \right)^2 \quad - (21)$$

where  $E = \gamma mc^2 \quad - (22)$

so 
$$p^2 + \kappa_0^2 = \frac{\gamma m}{\hbar^2} E \quad - (23)$$

Results

$$\begin{aligned} p^2 + \kappa_0^2 &= \frac{\gamma m}{\hbar^2} E \\ \kappa^2 + \kappa_0^2 &= \frac{\gamma m}{\hbar^2} (E - V) \end{aligned} \quad - (24)$$

Kinetic Energy Representation

In order to obtain a precise analogy with the Schrodinger equation, write eq. (9) as:

$$\frac{\hbar^2}{2m} \nabla^2 \psi = \frac{mc^2}{2} (\gamma^2 - 1) \psi \quad - (25)$$

In the non relativistic limit:

$$\frac{mc^2}{2} (\gamma^2 - 1) = \frac{mc^2}{2} \left( \left( 1 - \frac{v^2}{c^2} \right)^{-1} - 1 \right) \quad - (26)$$

$\xrightarrow{v \ll c} \frac{1}{2} mv^2$

5) So: 
$$\frac{\hat{p}^2}{2m} \psi = \frac{1}{2} m v^2 \psi \quad - (27)$$

For a free particle:

$$E = T = \frac{1}{2} m v^2 \quad - (28)$$

So 
$$\frac{\hat{p}^2}{2m} \psi = E \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi \quad - (29)$$

and

$$\nabla^2 \psi = -\left(\frac{2mE}{\hbar^2}\right) \psi \quad - (30)$$

i.e

$$(\nabla^2 + k^2) \psi = 0 \quad - (31)$$

where

$$k^2 = \frac{2mE}{\hbar^2} \quad - (32)$$

which is the non-relativistic limit of:

$$k^2 = \frac{2mE}{\hbar^2}; \quad E = \frac{mc^2}{2} (\gamma^2 - 1)$$

This is a required relativistic generalization of the  $k$  of the incoming particle in quantum tunnelling theory. - (33)

6) In the presence of a potential, eq. (25) becomes:

$$\left( \frac{\hbar^2 \nabla^2}{2m} + V \right) \psi = \frac{mc^2}{2} (\gamma^2 - 1) \psi \quad - (27)$$
$$= E \psi$$

So

$$-\hbar^2 \nabla^2 \psi = 2m(E - V) \psi \quad - (28)$$
$$= \kappa^2 \hbar^2 \psi$$

and

$$\kappa^2 = \frac{2m(E - V)}{\hbar^2}; \quad E = \frac{mc^2}{2} (\gamma^2 - 1)$$

- (29)  
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The relativistic transmission coefficient is:

$$= 16 \kappa^2 k^2 \left/ \left( \left( e^{4\kappa a} + e^{-4\kappa a} \right) (k^2 + \kappa^2) + 2(\kappa^4 + k^4 - 6\kappa^2 k^2) \right) \right. \quad - (30)$$

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with  $k$  and  $\kappa$  defined by eqs. (33) and (37).

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