226(9): Interaction of Two particles in Relativistic Physics

Consider two particles of four momenta p^μ and p_1^μ:

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \; ; \; p_1^\mu = \left(\frac{E_1}{c}, \underline{p}_1\right) \quad -(1)$$

In the semi classical development:

$$p^\mu = i\hbar \partial^\mu \quad -(2)$$

where

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) \quad -(3)$$

In the minimal prescription the interaction is described by:

$$p^\mu \rightarrow p^\mu + p_1^\mu, \quad -(4)$$

so

$$E \rightarrow E + E_1 \quad -(5)$$

$$\underline{p} \rightarrow \underline{p} + \underline{p}_1 . \quad -(6)$$

Here E is the total relativistic energy:

$$E = \gamma m c^2 \quad -(7)$$

and $\underline{p}$ is the relativistic momentum:

$$\underline{p} = \gamma m \underline{v}, \quad -(8)$$

where γ is the Lorentz factor:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} . \quad -(9)$$

Eq. (8) implies that:

$$E^2 = c^2 p^2 + m^2 c^4 . \quad -(10)$$

which is the Einstein energy equation. From eq. (10):

$$E^2 - m^2c^4 = (E - mc^2)(E + mc^2) = c^2p^2 \quad -(11)$$

so the relativistic kinetic energy T is:

$$T = E - mc^2 = (\gamma - 1)mc^2 = \frac{c^2p^2}{E + mc^2}. \quad -(12)$$

Therefore

$$T = \frac{\gamma^2c^2m^2v^2}{(\gamma+1)mc^2}, \quad -(13)$$

i.e.

$$\boxed{T = \left(\frac{\gamma^2}{1+\gamma}\right)mv^2} \quad -(14)$$

In the non-relativistic limit:

$$\gamma \rightarrow 1 \quad -(15)$$

so

$$T \rightarrow \frac{1}{2}mv^2 \quad -(16)$$

which is the non-relativistic kinetic energy.

From eqs. (4) and (10):

$$(E + E_1)^2 = c^2(p + p_1)^2 + m^2c^4 \quad -(17)$$

and so this is the result of the minimal prescription on the classical relativistic level. Therefore:

$$(E + E_1)^2 - m^2c^4 = c^2(p + p_1)^2. \quad -(18)$$

So:

$$E + E_1 - mc^2 = \frac{c^2 (p + p_1)^2}{E + E_1 + mc^2}$$

$$= \frac{c^2 m^2 (\gamma v + \gamma_1 v_1)^2}{mc^2 (1 + \gamma + \gamma_1)} \quad - (19)$$

The relativistic kinetic energy after interaction is:

$$\boxed{\begin{aligned} T_{int} &= E + E_1 - mc^2 \\ &= m \frac{(\gamma v + \gamma_1 v_1)^2}{1 + \gamma + \gamma_1} \end{aligned}} \quad - (20)$$

where:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad \gamma_1 = \left(1 - \frac{v_1^2}{c^2}\right)^{-1/2} \quad - (21)$$

Here v is the velocity of particle m and v_1 is the velocity of particle m_1.

These are the results of the theory without any approximation. The relativistic classical limit of the Dirac type approximations must now be considered to understand how the theoretical predictions are made of well known

+) phenomena such as the g factor of the electron, the Zeeman effect, the Landé factor, the anomalous Zeeman effect, the Thomas factor, spin orbit coupling, ESR, NMR, MRI and the Darwin term in spectroscopy. These approximations are tortuous and specially chosen to give the experimental results. They have never been properly tested outside the above phenomena.

The approximations start by writing eq. (17) as:

$$E + E_1 = \frac{c^2 (p + p_1)^2}{E + E_1} + \frac{m^2 c^4}{E + E_1}. \quad -(22)$$

Now add mc^2 to both sides:

$$E + E_1 + mc^2 = \frac{c^2 (p + p_1)^2}{E + E_1} + \frac{m^2 c^4}{E + E_1} + mc^2. \quad -(23)$$

The Dirac approximation assumes:

$$E_1 \ll E. \quad -(24)$$

So eq. (23) is written as:

$$E + E_1 + mc^2 = \frac{c^2 (p + p_1)^2}{E} + \frac{m^2 c^4}{E} + mc^2. \quad -(25)$$

The Dirac approximation then assumes:

5)

$$E = \gamma mc^2 \to mc^2 \quad -(26)$$

i.e. a particular type of non-relativistic limit. Furthermore this approximation is used in a carefully chosen way in eq. (25) to give:

$$2mc^2 + E_1 = \frac{c^2(p+p_1)^2}{E} + \frac{m^2c^4}{mc^2} + mc^2$$

$$= \frac{c^2(p+p_1)^2}{E} + 2mc^2 \quad -(26)$$

So a factor of 2 is very carefully chosen in this way. This is the g factor of the electron, Landé factor and Thomas factor.

The claims of the Dirac approximation all rest on this very careful choice of ~~approximation~~ method

Eq. (26) is rearranged as:

$$E = \frac{c^2(p+p_1)^2}{2mc^2+E_1} + \frac{2mc^2 E}{2mc^2+E_1} \quad -(27)$$

Finally it is assumed that:

$$E_1 << 2mc^2, \quad -(28)$$

So :

6)

$$E = \frac{c^2(p+p_1)^2}{2mc^2+E_1} + mc^2 \quad -(29)$$

where eq. (26) has been used. So:

$$T = E - mc^2 = \frac{c^2(p+p_1)^2}{2mc^2+E_1} \quad -(30)$$

$$= \frac{1}{2m}(p+p_1)^2\left(1+\frac{E_1}{2mc^2}\right)^{-1}$$

In the approximation (28):

$$\boxed{T = \frac{1}{2m}(p+p_1)^2\left(1-\frac{E_1}{2mc^2}\right)} \quad -(31)$$

Comparing eqs. (19) and (30) it is seen that eq. (19) has been approximated by use of eq. (26),

so eq. (19) becomes:

$$T = E + E_1 - mc^2 \sim \frac{c^2(p+p_1)^2}{2mc^2+E_1} \quad -(32)$$

and eq. (32) is approximated by eq. (31) using:

$$T = E + E_1 - mc^2 \sim E - mc^2 \quad -(33)$$

In order to quantize this theory the fermion equation is used:

7)

$$((E+E_1)+c\underline{\sigma}\cdot(\underline{p}+\underline{p}_1))\phi^L = mc^2\phi^R \quad -(34)$$

$$((E+E_1)-c\underline{\sigma}\cdot(\underline{p}+\underline{p}_1))\phi^R = mc^2\phi^L \quad -(35)$$

where

$$\phi^L = \begin{bmatrix}\psi_1^L \\ \psi_2^L\end{bmatrix}, \quad \phi^R = \begin{bmatrix}\psi_1^R \\ \psi_2^R\end{bmatrix} \quad -(36)$$

Therefore:

$$((E+E_1)^2 - c^2\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)\underline{\sigma}\cdot(\underline{p}+\underline{p}_1))\phi^L = m^2c^4\phi^L \quad -(37)$$

and similarly for ϕ^R.

The Dirac approximation is as follows:

$$(E+E_1)\phi^L = \left(\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)\left(\frac{c^2}{E+E_1}\right)\underline{\sigma}\cdot(\underline{p}+\underline{p}_1) + \frac{m^2c^4}{E+E_1}\right)\phi^L. \quad -(38)$$

Add mc^2 to each side:

$$(E+E_1+mc^2)\phi^L = \left(\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)\left(\frac{c^2}{E+E_1}\right)\underline{\sigma}\cdot(\underline{p}+\underline{p}_1) + \frac{m^2c^4}{E+E_1} + mc^2\right)\phi^L \quad -(39).$$

Following the same route of approximation as in the foregoing classical development, it is seen that eq. (39) reduces to:

$$(2mc^2+E_1)\phi^L = \left(\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)\frac{c^2}{E}\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)+\frac{m^2c^4}{E}+mc^2\right)\phi^L \quad -(40)$$

so:

$$E\phi^L = \left(\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)\left(\frac{c^2}{2mc^2+E_1}\right)\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)+\frac{m^2c^4+mc^2E}{2mc^2+E_1}\right)\phi^L \quad -(41)$$

The following approximations are now made:

$$\frac{m^2c^4+mc^2E}{(2mc^2+E_1)} \doteq mc^2 \quad -(42)$$

using:

$$E_1 \ll 2mc^2 \quad -(43)$$

$$E \sim mc^2 \quad -(44)$$

Therefore:

$$\boxed{\hat{H}\phi^L = T\phi^L} \quad -(45)$$

where:

$$T = E - mc^2 \quad -(46)$$

$$\hat{H} = \frac{1}{2m}\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)\left(1+\frac{E_1}{2mc^2}\right)^{-1}\underline{\sigma}\cdot(\underline{p}+\underline{p}_1), \quad -(47)$$

i.e.

$$\boxed{\hat{H} = \frac{1}{2m}\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)\left(1-\frac{E_1}{2mc^2}\right)\underline{\sigma}\cdot(\underline{p}+\underline{p}_1)} \quad -(48)$$

9) In eq. (48):

$$\boldsymbol{p} = -i\hbar \boldsymbol{\nabla} \quad -(49)$$

so

$$\hat{H} = \hat{H}_1 + \hat{H}_2 \quad -(50)$$

where:

$$\hat{H}_1 = \frac{1}{2m} \boldsymbol{\sigma} \cdot \left(-i\hbar \boldsymbol{\nabla} + \boldsymbol{p}_1\right) \boldsymbol{\sigma} \cdot \left(-i\hbar \boldsymbol{\nabla} + \boldsymbol{p}_1\right), \quad -(51)$$

$$\hat{H}_2 = -\boldsymbol{\sigma} \cdot \left(-i\hbar \boldsymbol{\nabla} + \boldsymbol{p}_1\right) \frac{E_1}{4m^2c^2} \left(-i\hbar \boldsymbol{\nabla} + \boldsymbol{p}_1\right). \quad -(52)$$

The $\hat{H}_1$ operator gives the g factor of the electron, the Landé factor, the anomalous Zeeman effect, and ESR, NMR, MRI and so on. The $\hat{H}_2$ operator gives the Thomas factor, spin-orbit effects in spectroscopy, and the Darwin term.

The algebra is worked out as follows, and is illustrated with eq. (51):

$$\boldsymbol{\sigma} \cdot (\boldsymbol{p} + \boldsymbol{p}_1) \boldsymbol{\sigma} \cdot (\boldsymbol{p} + \boldsymbol{p}_1) = (\boldsymbol{p} + \boldsymbol{p}_1) \cdot (\boldsymbol{p} + \boldsymbol{p}_1) + i\boldsymbol{\sigma} \cdot (\boldsymbol{p} + \boldsymbol{p}_1) \times (\boldsymbol{p} + \boldsymbol{p}_1)$$

$$= \boldsymbol{p} \cdot \boldsymbol{p} + \boldsymbol{p}_1 \cdot \boldsymbol{p} + \boldsymbol{p} \cdot \boldsymbol{p}_1 + \boldsymbol{p}_1 \cdot \boldsymbol{p}_1 + i\boldsymbol{\sigma} \cdot (\boldsymbol{p} \times \boldsymbol{p} + \boldsymbol{p}_1 \times \boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{p}_1 + \boldsymbol{p}_1 \times \boldsymbol{p}_1)$$

$$= \underline{p} \cdot \underline{p} + \underline{p}_1 \cdot \underline{p} + \underline{p} \cdot \underline{p}_1 + \underline{p}_1 \cdot \underline{p}_1 \quad -(53)$$
$$+ i \underline{\sigma} \cdot (\underline{p}_1 \times \underline{p} + \underline{p} \times \underline{p}_1).$$

So:

$$\boxed{\hat{H}_1 = -\frac{\hbar^2}{2m} \nabla^2 + \frac{p_1^2}{2m} + \frac{i\hbar}{2m} (\underline{p}_1 \cdot \underline{\nabla} + \underline{\nabla} \cdot \underline{p}_1) + \frac{\hbar}{2m} \underline{\sigma} \cdot (\underline{p}_1 \times \underline{\nabla} + \underline{\nabla} \times \underline{p}_1)} \quad -(54)$$

This hamiltonian operator gives a series of kinetic energy eigenvalues:

$$\hat{H}_1 \phi^L = T \phi^L \quad -(55)$$

and these are all relativistic kinetic energies.

The algebra is worked out as follows:

$$(\underline{p}_1 \cdot \underline{\nabla}) \phi^L = \underline{p}_1 \cdot \underline{\nabla} \phi^L \quad -(56)$$

$$(\underline{\nabla} \cdot \underline{p}_1) \phi^L = \underline{\nabla} \cdot (\underline{p}_1 \phi^L) \quad -(57)$$
$$= (\underline{\nabla} \cdot \underline{p}_1) \phi^L + \underline{p}_1 \cdot \underline{\nabla} \phi$$

using the Leibnitz Theorem. Similarly:

$$(\underline{p}_1 \times \underline{\nabla}) \phi^L = \underline{p}_1 \times \underline{\nabla} \phi^L \quad -(58)$$

and:

11)

$$\underline{\nabla} \times (\underline{p}_1 \phi^L) = (\underline{\nabla} \times \underline{p}_1) \phi^L + \underline{\nabla} \phi^L \times \underline{p}_1 \quad -(59)$$

Note that:

$$\underline{p}_1 \times \underline{\nabla} \phi^L + \underline{\nabla} \phi^L \times \underline{p}_1 = \underline{0} \quad -(60)$$

Therefore:

$$\boxed{\begin{aligned} \hat{H}_1 = & -\frac{\hbar^2}{2m} \nabla^2 + \frac{p_1^2}{2m} \\ & + \frac{i\hbar}{2m} \left(\underline{\nabla} \cdot \underline{p}_1 + 2 \underline{p}_1 \cdot \underline{\nabla} \right) \\ & + \frac{\hbar}{2m} \underline{\sigma} \cdot \underline{\nabla} \times \underline{p}_1 \end{aligned}} \quad -(61)$$

and

$$\boxed{\hat{H}_1 \phi^L = T \phi^L} \quad -(62)$$

This general theoretical structure may be applied now to a large number of problems.

For example, if the interaction of an electron with the electromagnetic field is considered, it is possible to make the minimal prescription as:

$$p^\mu \rightarrow p^\mu + eA^\mu \quad -(63)$$

on the u(1) level merely for the sake of illustration.

On the ECE level:

$$p^a_\mu \rightarrow p^a_\mu + eA^a_\mu . \quad -(64)$$

and this will be developed in future work. We may use:

$$eA^\mu = \hbar \kappa^\mu \quad -(65)$$

to describe photon absorption. Here:

$$A^\mu = \left(\frac{\phi}{c}, \underline{A} \right) \quad -(66)$$

$$\kappa^\mu = \left(\frac{\omega}{c}, \underline{\kappa} \right) \quad -(67)$$

So:

$$\boxed{\hat{H}_1 = -\frac{\hbar^2}{2m} \nabla^2 + \frac{e^2 A^2}{2m} + \frac{ie\hbar}{2m} \left(\underline{\nabla} \cdot \underline{A} + 2 \underline{A} \cdot \underline{\nabla} \right) + \frac{e\hbar}{2m} \underline{\sigma} \cdot \underline{\nabla} \times \underline{A}} \quad -(68)$$

On the u(1) level the magnetic flux density in tesla is:

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad -(69)$$

so the ESR term is

$$\hat{H}_{ESR} = \frac{e\hbar}{2m} \underline{\sigma} \cdot \underline{B} \quad -(70)$$