226(6): Development from the Classical Relativistic Level

Start from the Einstein energy equation for a free particle:

$$E^2 = (\gamma m c^2)^2 = c^2 p^2 + m^2 c^4 \quad -(1)$$

where:

$$\underline{p} = \gamma m \underline{v}, \quad -(2)$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(3)$$

Then:

$$E = \gamma m c^2 = \frac{E^2}{\gamma m c^2} \quad -(4)$$

$$E = \frac{1}{\gamma m c^2}\left(c^2 p^2 + m^2 c^4\right)$$

$$= \frac{p^2}{\gamma m} + \frac{m c^2}{\gamma}.$$

So

$$\boxed{E = \gamma m v^2 + \frac{m c^2}{\gamma}} \quad -(5)$$

and

$$(E^2 - c^2 p^2) = m^2 c^4. \quad -(6)$$

The basic postulate of quantum mechanics is:

$$\hat{p}^\mu = i\hbar \partial^\mu \quad -(7)$$

where

$$\hat{p}^\mu = \left(\frac{\hat{E}}{c}, \hat{\underline{p}}\right) - (8)$$

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) - (9)$$

2) So:

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \quad \hat{\underline{p}} = -i\hbar \underline{\nabla} \quad -(10)$$

Let the particle wave function be ϕ. Then eq. (5) is:

$$\hat{E}\phi = \left(\gamma m v^2 + \frac{mc^2}{\gamma}\right)\phi = E\phi. \quad -(11)$$

This is an expression of the fermion equation. Eq. (11) means that:

$$i\hbar \frac{\partial \phi}{\partial t} = E\phi \quad -(12)$$

i.e

$$\frac{\partial \phi}{\partial t} = -\frac{iE}{\hbar}\phi, \quad -(13)$$

a solution of which is:

$$\boxed{\phi = \exp\left(-\frac{iE}{\hbar}t\right)} \quad -(14)$$

i.e.

$$\boxed{\phi = \exp\left(-\frac{i}{\hbar}\left(\gamma m v^2 + \frac{mc^2}{\gamma}\right)t\right)} \quad -(15)$$

This is the wave function of a free particle in relativistic quantum mechanics.

3)

The wave function of a particle at rest is:

$$\phi_0 = \exp\left(-\frac{i m c^2}{\hbar} t \right) \quad -(16)$$

so in relativistic quantum mechanics a particle at rest has energy:

$$E_0 = mc^2 \quad -(17)$$

and oscillates.

On the classical level the relativistic kinetic energy is:

$$T = E - mc^2 = (\gamma - 1) mc^2. \quad -(18)$$

So:

$$\boxed{T = \gamma m v^2 + \frac{mc^2}{\gamma} - mc^2} \quad -(19)$$

i.e

$$T = \gamma m v^2 + mc^2 \left(\frac{1}{\gamma} - 1 \right)$$

$$T = \gamma m v^2 + mc^2 \left(\frac{1 - \gamma}{\gamma} \right) \quad -(20)$$

It can be written as:

$$T = \gamma m v^2 + mc^2 \left(1 - \frac{v^2}{c^2} \right)^{1/2} - mc^2 \quad -(21)$$

4) In the limit:

$$v << c \quad -(21)$$

$$T \to \gamma m v^2 + mc^2\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) - mc^2$$

$$= \gamma m v^2 - \frac{1}{2} m v^2 \quad -(22)$$

In the limit (21):

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \to 1 \quad -(23)$$

so

$$T \to \frac{1}{2} m v^2 = \frac{p^2}{2m} \quad -(24)$$

Self consistently, the same result is obtained from eq. (18):

$$T = (\gamma - 1) mc^2 \quad -(25)$$

$$= \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right) mc^2$$

$$\to \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) mc^2$$

$$= \frac{1}{2} m v^2 = \frac{p^2}{2m}$$

Therefore:

5)
$$T = E - mc^2 \rightarrow \frac{p^2}{2m} \quad -(26)$$

In the operator representation:
$$\frac{\hat{p}^2}{2m}\phi = T\phi = (E - mc^2)\phi \quad -(27)$$

i.e
$$-\frac{\hbar^2 \nabla^2}{2m}\phi = T\phi = (E - mc^2)\phi \quad -(28)$$

This limit is denoted the Schroedinger equation:
$$\hat{H}\phi = T\phi \quad -(29)$$

Minimal Prescription

Usually this is introduced via considering the interaction of a fermion with the electromagnetic field:
$$p^\mu \rightarrow p^\mu - eA^\mu \quad -(30)$$

where
$$A^\mu = \left(\frac{\phi}{c}, \underline{A}\right) \quad -(31)$$

So:
$$E \rightarrow E - e\phi, \quad \underline{p} \rightarrow \underline{p} - e\underline{A} \quad -(32)$$

6) In the ECE theory, eq. (33) is used for each polarization index a.

So eq. (1) becomes:

$$(E - e\phi)^2 = c^2 (p - eA)^2 + m^2 c^4 \quad -(33)$$

i.e.

$$E - e\phi = \frac{c^2 (p - eA)^2}{E - e\phi} + \frac{m^2 c^4}{E - e\phi} \quad -(34)$$

It is convenient to write this equation as:

$$E - e\phi = c^2 (p - eA)(E - e\phi)^{-1}(p - eA) + \frac{m^2 c^4}{E - e\phi} \quad -(35)$$

or as:

$$E - e\phi - \frac{c^2 (p - eA)^2}{E - e\phi} = \frac{m^2 c^4}{E - e\phi} \quad -(36)$$

The g factor of the electron, the Thomas factor and Darwin term emerge from a particular approximation of the quantized form of eq. (36).

First add mc^2 to both sides of eq. (36):

7)

$$\boxed{E - e\phi + mc^2 - \frac{c^2 (p - eA)^2}{E - e\phi} = \frac{m^2 c^4}{E - e\phi} + mc^2}$$

— (37)

This equation is the result of the minimal prescription on the classical relativistic level.

Now use the approximation:

$$e\phi << E \quad - (38)$$

in the denominators of eq. (37), so:

$$E - e\phi + mc^2 - \frac{c^2}{E}(p - eA)^2 = \frac{m^2 c^4}{E} + mc^2$$

— (39)

The second approximation used is:

$$E = \gamma mc^2 \rightarrow mc^2 \quad - (40)$$

so:

$$2mc^2 - e\phi - \frac{c^2}{E}(p - eA)^2 = 2mc^2$$

— (41)

This is written as:

$$(2mc^2 - e\phi)E = c^2(p - eA)^2 + 2mc^2 E \quad -(42)$$

i.e.

$$E = c^2(p - eA)(2mc^2 - e\phi)^{-1}(p - eA) + \frac{2mc^2}{2mc^2 - e\phi} E \quad -(43)$$

Quantization is introduced as follows:

$$\hat{E}\phi = \left(c^2(p - eA)(2mc^2 - e\phi)^{-1}(p - eA) + \frac{2mc^2 E}{2mc^2 - e\phi} \right)\phi \quad -(44)$$

where $\hat{E}$ on the left hand side is an operator and E on the right hand side is as:

$$E = \gamma mc^2 \quad -(45)$$

In the approximations (38) and (40) eq. (44) becomes:

$$\hat{E}\phi = \left(\frac{1}{2m}(\underline{p} - e\underline{A})\left(1 - \frac{e\phi}{2mc^2}\right)^{-1}(\underline{p} - e\underline{A}) + mc^2 \right)\phi \quad -(45)$$

9) The extra factor 2 in eq. (45) give the g factor of the electron and the Thomas precession. This very special approximation procedure is justified only by the experimental results.

Finally the Pauli matrices are introduced as follows:

$$\hat{E}\phi = \left(\frac{1}{2m}\left(\underline{\sigma}\cdot(\underline{p}-e\underline{A})\right)\left(1-\frac{e\phi}{2mc^2}\right)^{-1}\left(\underline{\sigma}\cdot(\underline{p}-e\underline{A})\right) + mc^2\right)\phi \quad -(46)$$

and

$$\left(1-\frac{e\phi}{2mc^2}\right)^{-1} \sim 1 + \frac{e\phi}{2mc^2} \quad -(47)$$

so:

$$\left(\hat{E}-mc^2\right)\phi = \frac{1}{2m}\left(\underline{\sigma}\cdot(\underline{p}-e\underline{A})\left(1+\frac{e\phi}{2mc^2}\right)\underline{\sigma}\cdot(\underline{p}-e\underline{A})\right)\phi \quad -(48)$$

$$= \frac{1}{2m}\underline{\sigma}\cdot(\underline{p}-e\underline{A})\,\underline{\sigma}\cdot(\underline{p}-e\underline{A})\phi + \underline{\sigma}\cdot(\underline{p}-e\underline{A})\frac{e\phi}{4m^2c^2}\underline{\sigma}\cdot(\underline{p}-e\underline{A})\phi \quad -(49)$$

10) The first term on the RHS gives the g factor of the electron, ESR, NMR and MRI etc., and the second term gives spin-orbit coupling and the Darwin term. These emerge by a second method of quantization from eq. (43) as follows: Approximate eq. (43) by:

$$E = \frac{1}{2m}(p-eA)\left(1-\frac{e\phi}{2mc^2}\right)^{-1}(p-eA) + mc^2, \quad -(50)$$

so:

$$\boxed{\frac{1}{2m}(p-eA)\left(1-\frac{e\phi}{2mc^2}\right)^{-1}(p-eA) = E - mc^2 = T} \quad -(51)$$

The kinetic energy is therefore approximately:

$$\boxed{T = \frac{1}{2m}(p-eA)\left(1+\frac{e\phi}{2mc^2}\right)(p-eA)} \quad -(52)$$

i.e. $$T = \frac{1}{2m}(p-eA)^2 + (p-eA)\frac{e\phi}{4m^2c^2}(p-eA) \quad -(53)$$

In the σ basis:

$$T = \frac{1}{2m} \underline{\sigma} \cdot (\underline{p} - e\underline{A}) \underline{\sigma} \cdot (\underline{p} - e\underline{A})$$
$$+ \underline{\sigma} \cdot (\underline{p} - e\underline{A}) \frac{e\phi}{4m^2c^2} \underline{\sigma} \cdot (\underline{p} - e\underline{A}) \quad -(54)$$

Quantization is introduced by:

$$\hat{p} = -i\hbar \underline{\nabla} \quad -(55)$$

So:

$$T\phi = \Big(\frac{1}{2m} \underline{\sigma} \cdot (-i\hbar\underline{\nabla} - e\underline{A}) \underline{\sigma} \cdot (-i\hbar\underline{\nabla} - e\underline{A})$$
$$+ \underline{\sigma} \cdot (-i\hbar\underline{\nabla} - e\underline{A}) \frac{e\phi}{4m^2c^2} \underline{\sigma} \cdot (-i\hbar\underline{\nabla} - e\underline{A}) \Big) \phi \quad -(56)$$

This structure is obtained from the fermion eqn.

$$\boxed{\begin{aligned} \left((E + e\phi) + c\underline{\sigma} \cdot (\underline{p} + e\underline{A}) \right) \phi^L &= mc^2 \phi^R \\ \left((E + e\phi) - c\underline{\sigma} \cdot (\underline{p} + e\underline{A}) \right) \phi^R &= mc^2 \phi^L \end{aligned}} \quad -(57)$$

and can be used for any type of particle interaction.