175(7): Development of $[x^2, \hat{p}^2]\phi$.

Using the commutator equation:

$$[A, BC] = [A, B]C + B[A, C] \quad -(1)$$

it is seen that:

$$[x^2, \hat{p}^2] = [x^2, \hat{p}\hat{p}] = [x^2, \hat{p}]\hat{p} + \hat{p}[x^2, \hat{p}] \quad -(2)$$

In general:

$$p^2 = p_x^2 + p_y^2 + p_Z^2. \quad -(3)$$

Now use:

$$[p, x^2]\phi = -2i\hbar x \quad -(4)$$

$$[x^2, \hat{p}]\phi = 2i\hbar x \quad -(5)$$

It follows that:

$$\boxed{[x^2, \hat{p}^2]\phi = 2i\hbar(x\hat{p} + \hat{p}x)\phi} \quad -(6)$$

where the anti-commutator is defined by:

$$\{x, \hat{p}\}\phi = (x\hat{p} + \hat{p}x)\phi \quad -(7)$$

So:

$$\boxed{[x^2, \hat{p}^2]\phi = 2i\hbar\{x, \hat{p}\}\phi} \quad -(8)$$

By definition:

$$\{x, \hat{p}\}\phi = -i\hbar\left(x\frac{\partial\phi}{\partial x} + \frac{\partial}{\partial x}(x\phi)\right)$$

$$= -i\hbar\left(x\frac{\partial\phi}{\partial x} + x\frac{\partial\phi}{\partial x} + \phi\right)$$

$$= -2i\hbar x\frac{\partial\phi}{\partial x} - i\hbar\phi \quad -(9)$$

Therefore:

$$[x^2, \hat{p}^2]\phi = [x^2, \hat{p}\hat{p}]\phi$$

$$= 2\hbar^2\phi + 4\hbar x\frac{\partial\phi}{\partial x} \quad -(10)$$

$$= 2\hbar^2\phi + 4i\hbar x\hat{p}$$

which is the same result as in note 175 (4).

Eq. (8) is a new general result of quantum mechanics, introducing a relation between the commutator and anti-commutator.

Numerical results so far show that the anti-commutator $\{x, \hat{p}\}\phi$ vanishes for all the wavefunctions of the harmonic oscillator and the particle in a box.

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